

# THE CHIRAL 2-SPHERE

by

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## Abstract

The two dimensional surface of a sphere can be parametrized by coordinates representing two charged pions acting as Goldstone bosons of a broken  $SU_2$  symmetry. We construct in full concrete detail, and in a general class of coordinate systems, all the relevant structure forming a framework for this low energy effective theory.

# 1 Introduction

It is now some 25 years since non-linear chiral  $SU_2 \times SU_2$  Lagrangians were introduced to study the experimental consequences of the emergence of three massless pions as Goldstone bosons, and the results have been clearly exhibited in excellent review articles [1, 2]. Later a very detailed and remarkably successful effective chiral Lagrangian perturbative treatment of low energy physics was proposed by Gasser and Leutwyler [3, 4] and is now regarded as standard in the field. In such schemes the transformations of the Goldstone bosons are non-linear and general treatments of the required coset-space mathematics are well established and elegant in form [5, 6]. Also, the consequential construction of invariant non-linear Lagrangians is standard and well known [7, 8].

From time to time, as in the case of effective chiral Lagrangians mentioned above, there are developments in physics which create a resurgence of interest in the structure. This was particularly the case when supersymmetric  $\sigma$  models were first taken seriously [9, 10] because of similarities of their properties in two dimensions with the structure of four-dimensional gauge theories [11]. The generalization to  $CP_N$  models in four dimensions [12] followed swiftly, and a seminal paper by Zumino [13] showed the central place of geometry in the models, with the Kähler metric of complex manifolds providing an elegant description of the supersymmetry. There then followed a decade in which the main focus of attention was on preon like models in which the dominant theme was that the supersymmetry helped to ensure the existence of light fermions by relating them to bosons which were in turn kept light by the Goldstone theorem. A general analysis of the required features can be obtained by working backwards through the literature from the references given in papers by Kotcheff and Shore [14], and by Buchmuller and Lerche [15], both of which are written with authority and also have fine introductory sections.

Recently there have been two developments which suggest a yet further resurgence of interest in these topics. The electric-magnetic duality conjectured by Olive and Montonen [16] several years ago, and shown by Osborn [17] to be related to  $N = 4$  supersymmetric gauge theories, has emerged in a generalization in the work of Sen [18]. Moreover, this seems to play an important role in the work of Seiberg and Witten [19] involving  $N = 2$  supersymmetric gauge theory in four dimensions. On an apparently unrelated front, following the emergence of the Supersymmetric Standard Model as a major candidate for physics beyond the Standard Model, has come the realization that supersymmetry may appear in nature at energies which may soon be experimentally accessible. Thus a supersymmetric extension of chiral perturbation theory becomes of real interest. Already, two attempts have

been made in this direction [20, 21] both based on linear supersymmetric models in which the symmetry is broken (but the supersymmetry preserved) as the Higgs mass becomes infinite.

It seems that supersymmetric sigma models are ripe for further investigation, and obviously the simplest underlying Kähler manifold is the 2-sphere [22]. What is presented in this paper is a direct treatment of the manifold structure, the nonlinear transformation laws of the Goldstone bosons, and the construction of the invariant Lagrangians, all in a general class of coordinate systems. Curiously, although the 2-sphere has been much studied this does not seem to have been recorded before. There are, of course, versions in coordinates resulting from constrained linear  $\sigma$  models, treatments in exponential (standard) coordinates, projective coordinate presentations, and most importantly stereographic coordinate representations revealing the Kähler structure. Our general coordinate treatment includes and relates all of these, and we believe it reveals the structure in much the way that covariant notation clarifies special relativity.

We shall show how this model, although not physical, is uniquely embedded in chiral  $SU_2 \times SU_2$  (which is indeed of direct physical interest as noted above) and retains many of the relevant features thus allowing them to be studied in a much simpler and concrete way. It is a very useful theoretical laboratory.

One of the primary motivations for the current presentation, it has to be admitted, is the experience of one of the authors (K.J.B.) who over many years has persisted in quoting some of the results as “self-evident” consequences of the embedding in chiral  $SU_2 \times SU_2$  which is fully analysed [23]. Only repeated objections of friends and colleagues have finally persuaded him that “obvious” is not equivalent to mathematically proved. This paper shows that even though the techniques used in reference [23] are not all valid in the present case, nevertheless the results obtained in the embedded limit are identical.

## 2 The Chiral Sphere

We start this section by reviewing [23] the structure of chiral  $SU_2 \times SU_2$  to establish notation. The transformation of the fundamental (quark) multiplet is specified by

$$q \rightarrow q - i\theta_i \frac{\tau^i}{2} q - i\phi_i \frac{\tau^i}{2} (i\gamma_5) q \quad (1)$$

to lowest order in the real parameters  $\theta_i$  and  $\phi_i$ ,  $1 \leq i \leq 3$ , where  $\tau^i$  are the familiar Pauli matrices. Note the extra  $(i\gamma_5)$  factors in the final terms which are included to ensure that the Goldstone bosons of this scheme will be pseudoscalar. The crucial step in describing

these bosons is to parametrize the coset space defined by the quotient of the  $SU_2 \times SU_2$  by the vector  $SU_2$  parameterised by the  $\theta^i$  alone. This takes the simple form

$$\hat{L} = \exp \left\{ \frac{-i\theta}{2} n_i \tau^i (i\gamma_5) \right\} \quad (2)$$

where the Goldstone fields are described by

$$M^i = M n^i, \quad (3)$$

with

$$(n^i)^2 = 1, \quad (4)$$

so that

$$(M^i)^2 = M^2, \quad (5)$$

and  $\theta$  is an arbitrary function of  $M$ . This arbitrariness may be viewed as the freedom to change coordinate systems on the coset space, or to redefine the field variables describing the mesons. If we define projection operators by

$$P_L = \frac{1}{2}(1 + i\gamma_5), \quad (6)$$

and

$$P_R = \frac{1}{2}(1 - i\gamma_5), \quad (7)$$

so that

$$P_L P_L = P_L, \quad (8)$$

$$P_R P_R = P_R, \quad (9)$$

$$P_L P_R = 0 = P_R P_L, \quad (10)$$

and

$$P_L + P_R = 1, \quad (11)$$

then we can rewrite equation (2) as

$$\hat{L} = L P_L + L^{-1} P_R, \quad (12)$$

where  $L$  is unitary and the  $\gamma_5$  dependence is now contained solely in the projection operators. It is then clear that we can deal with

$$L = \exp \left\{ \frac{-i\theta}{2} n_i \tau^i \right\} \quad (13)$$

and reinstate the  $\gamma_5$  factors only when wishing to consider the explicit couplings of the Goldstone bosons to matter fields. The action of a group element  $g$  (of  $SU_2 \times SU_2$ ) on the coset space can be specified by

$$gL = L'h \quad (14)$$

where

$$L'(M_i) = L(M'_i), \quad (15)$$

specifies the non-linear transformations of the Goldstone boson fields,

$$h = \exp \left\{ \frac{-i}{2} \lambda_i \tau^i \right\}, \quad (16)$$

and the  $\lambda_i$  depend on the fields and the group parameters. What we have are non-linear transformations among the  $M_i$  (which give a realization of the group) which are linear under the action of the  $SU_2$  subgroup, thus neatly describing a situation where the full group is still realized, but in a manner well suited to spontaneous breaking to the subgroup. The Goldstone bosons are a linear representation of the  $SU_2$  subgroup only. Although the procedure extends to other representations, for our present purposes it will be sufficient to stay mostly in the fundamental representation.

We are now ready to discuss the chiral  $SU_2$  structure embedded in this framework. Consider the subgroup of the chiral  $SU_2 \times SU_2$  group specified in equation (1) by retaining only the parameters  $\theta_3$  and  $\phi_A$ , with  $A = 1$  and  $2$ . Obviously this is an  $SU_2$  subgroup, and we call it chiral  $SU_2$  in recognition of the  $(i\gamma_5)$  factors with the  $\tau^A$  generators. Clearly the  $\tau^3$  generates a  $U_1$  subgroup, so that the coset space obtained by the quotient of chiral  $SU_2$  by this  $U_1$  is parametrized by coordinates  $M_A$ ,  $A = 1$  and  $2$ , which can be viewed as describing two Goldstone pseudoscalars. Notice that the embedding of this  $SU_2/U_1$  structure in the  $\frac{SU_2 \times SU_2}{SU_2}$  structure is uniquely specified. Moreover, if we set  $M_3$  and  $n_3$  to zero in our previous discussion, then

$$L = \exp \left\{ \frac{-i\theta}{2} n_A \tau^A \right\}, \quad (17)$$

and set  $\lambda_A = 0$ , so

$$h = \exp \left\{ \frac{-i}{2} \lambda_3 \tau_3 \right\}, \quad (18)$$

where  $\theta$  is now an arbitrary function of

$$M^2 = M_1^2 + M_2^2 \quad (19)$$

which when  $M_3$  becomes zero remains as the only independent scalar.

We can now see the advantages of using this chiral 2-sphere as a model. It is simpler than the chiral  $SU_2 \times SU_2$  scheme even in the purely bosonic sector. Moreover, the 2-sphere

is a Kähler manifold and so admits a supersymmetric extension in which the Goldstone bosons acquire fermionic (Weyl) partners without yet more quasi-Goldstone bosons and fermions being forced into the model [22]. Also the resulting couplings among the particles are uniquely specified. Contrast this with the situations in references [20] and [21] where the number of bosons doubles, as does the number of associated fermions, and finally the couplings involving these new particles are not uniquely specified. Of course, these latter cases are closer to the physics of the real world (they have 3 pions for example), but the embedded chiral 2-sphere model retains many significant features and is a far more tractable theoretical laboratory. We now present the details of this model.

First we establish the transformation laws of the Goldstone fields under chiral  $SU_2$ . It is sufficient to work to lowest order in the group parameters and we denote the transformations by

$$g : M_A \rightarrow M_A + \theta_3 K_{3A} + \phi_B K_{BA}, \quad (20)$$

where  $K_{3A}$  and  $K_{BA}$  are Killing field components constructed from the  $M_A$  themselves. Of course, the action under an element of the  $U_1$  subgroup is linear so that  $K_{3A}$  is already known, but we shall let this emerge from our calculations. Expanding equation (14) we see that we need to solve

$$\begin{aligned} & \left[ 1 - \frac{i\theta_3}{2}\tau_3 - \frac{i\phi_B}{2}\tau_B \right] L(M) \\ &= [L(M) + L_{,A}\theta_3 K_{3A} + L_{,A}\phi_B K_{BA}] \times \left[ 1 - \frac{i}{2}\theta_3\tau_3 - \frac{i\phi_A\lambda_{A3}}{2}\tau_3 \right], \end{aligned} \quad (21)$$

where

$$L_{,A} = \frac{\partial L(M)}{\partial M_A}, \quad (22)$$

$$\lambda_3 = \theta_3 + \phi_A \lambda_{A3}, \quad (23)$$

and we note that in this particular simple example raising and lowering of indices is of no consequence if we preserve the order of indices on the Killing vector fields. It is clear that the calculations require nothing more than the construction of functions of Pauli matrices, but even so a little technique can be helpful. The quantities

$$P^\pm = \frac{1}{2}(1 \pm n_A \tau_A) \quad (24)$$

share the projection operator properties given in equations (8) to (11) for the  $P_L$  and  $P_R$ , as can easily be seen because the  $n_A$  form a unit vector. This means that equation (17) can be expressed as

$$L = P^+ \exp\left(\frac{-i\theta}{2}\right) + P^- \exp\left(\frac{i\theta}{2}\right), \quad (25)$$

and other functions can be similarly handled. Also, from equation (19) we see that

$$M_{,A} = n_A, \quad (26)$$

and differentiating

$$M_A = M n_A \quad (27)$$

yields

$$M n_{A,B} = \delta_{AB} - n_A n_B, \quad (28)$$

so that

$$\begin{aligned} P_{,B}^{\pm} &= \pm \frac{1}{2M} \tau_A (\delta_{AB} - n_A n_B) \\ &= \pm \frac{1}{2M} (\tau_B + n_B P^- - n_B P^+). \end{aligned} \quad (29)$$

We note that again the tensors  $(\delta_{AB} - n_A n_B)$  and  $n_A n_B$  have the by now familiar projection operator properties, so that calculations become systematic and straightforward. A little simple algebra applied to equation (21) reveals that

$$K_{BC} = M \cot \theta (\delta_{BC} - n_B n_C) + n_B n_C \frac{dM}{d\theta} \quad (30)$$

and

$$K_{3C} = \varepsilon_{3BC} n_B \phi = \varepsilon_{3BC} M_B, \quad (31)$$

where  $\varepsilon_{3BC}$  is the familiar totally antisymmetric Levi-Civita tensor. As noted previously  $K_{3C}$  is linear in the  $M_C$ , and we recognise the usual rotational transformation of a vector.

We have already found the transformation laws for the Goldstone bosons and, as the reader can easily check, these are identical to those given in reference [23] when the truncation of variables described above is applied. Returning to equations (14) and (18) we note, following reference [5], that if  $\psi$  is an irreducible representation of the unbroken subgroup, so that here (keeping to the fundamental representation) we have simply that

$$\psi \rightarrow \psi - \frac{i\theta_3}{2} \tau_3 \psi \quad (32)$$

then under the full group action

$$\psi \rightarrow \psi - \frac{i\theta_3}{2} \tau_3 \psi - \frac{i\phi_B \lambda_{B3}}{2} \tau_3 \psi \quad (33)$$

where

$$\lambda_{B3} = \varepsilon_{BA3} M_A \tan(\theta/2)/M. \quad (34)$$

Note that this transformation law is linear in  $\psi$ , but with non-linear coefficients constructed from  $M_A$ ; it and its generalizations are known as Standard Field transformations, and these

exhaust all field types. Again the reader can easily check that the result in equation (34) follows trivially from the corresponding result in reference [23] when our truncation method is applied.

What remains is to show how to construct invariant Lagrangians from the fields we have introduced. It is at this point that the objections (mentioned previously in the Introduction) arise to the direct extraction of further results from reference [23] by our truncation method. The difficulty is that later results in reference [23] explicitly use a property that is not available in the chiral  $SU_2$  substructure. In the full chiral  $SU_2 \times SU_2$  the Killing vectors can be combined into so called left and right combinations which viewed as matrices  $(K^L)_{AB}$  and  $(K^R)_{AB}$  are non-singular and can be inverted. Unfortunately, in the chiral  $SU_2$  substructure only  $K_{AB}$  and  $K_{3C}$  exist so that this trick (which is a useful shortcut) is not directly available. However, as we shall see,  $K_{AB}$  itself is non-singular, and by a slight extension of the calculations we do eventually reach the same results.

So what invariants can be constructed? This question was answered elegantly in reference [7]. The first point is that no invariant can be constructed from the  $M_A$  alone. In particular this implies that an invariant mass term is not available for the Goldstone bosons in accordance with the Goldstone theorem. Now consider derivatives of the fields. The key concept is found by rewriting equation (14) in the forms

$$L' = gLh^{-1}, \quad (35)$$

and

$$L'^{-1} = hL^{-1}g^{-1}, \quad (36)$$

and differentiating the former to obtain

$$\partial_\mu L' = g \left[ (\partial_\mu L) h^{-1} + L (\partial_\mu h^{-1}) \right], \quad (37)$$

where  $\partial_\mu = \frac{\partial}{\partial x^\mu}$  differentiates the fields  $M_A$ , but  $g$  is constant because we are considering only global transformations. From equations (36) and (37) we see

$$\begin{aligned} L^{-1} (\partial_\mu L) &\rightarrow L'^{-1} (\partial_\mu L') \\ &= h \left[ L^{-1} (\partial_\mu L) \right] h^{-1} + h (\partial_\mu h^{-1}) \end{aligned} \quad (38)$$

and recognise that, because  $h$  is in the subgroup, the transformation does not mix the coset space and subgroup generators in the algebra. Thus, if we write

$$\begin{aligned} 2iL^{-1} (\partial_\mu L) &= \tau_B a_\mu^B + \tau_3 v_\mu^3 \\ &= a_\mu + v_\mu \end{aligned} \quad (39)$$



then equation (38) gives

$$a_\mu \rightarrow h a_\mu h^{-1} \quad (40)$$

and

$$\begin{aligned} v_\mu &\rightarrow h v_\mu h^{-1} + h \left( \partial_\mu h^{-1} \right) \\ &= v_\mu + h \left( \partial_\mu h^{-1} \right) \end{aligned} \quad (41)$$

where the final simplification in equation (41) follows because the subgroup is abelian. It follows from equation (40) that the quantity

$$\frac{1}{2} \text{Tr} \left[ a_\mu^B a^\mu_B \right]$$

is an invariant, and in fact this is the only invariant which can be made from the Goldstone bosons which involves exactly two derivatives. Usually the notation of a covariant derivative

$$\Delta_\mu M^B = a_\mu^b \quad (42)$$

is introduced and the expression

$$\mathcal{L} = \frac{1}{2} \text{Tr} [(\Delta_\mu M^B)(\Delta^\mu M_B)] \quad (43)$$

written for the Lagrangian which has a leading order expansion in fields appropriate for interpretation as a kinetic energy term. Isham [8] introduced the metric form

$$\mathcal{L} = \frac{1}{2} g_{AB} (\partial_\mu M^A)(\partial^\mu M^B) \quad (44)$$

for this Lagrangian, thus giving a geometric understanding in terms of the metric  $g_{AB}$  on the coset space manifold. We return briefly to equation (41) to note that if there is a matter field  $\psi$  which transforms under the  $U_1$  subgroup so that

$$\psi \rightarrow \psi - \frac{i}{2} \theta_3 \tau^3 \psi \quad (45)$$

then reference [5] shows that under the full action of the chiral  $SU_2$

$$\psi \rightarrow \psi - \frac{i\theta_3}{2} \tau_3 \psi - \frac{i\lambda_{A3}\phi_A}{2} \tau_3 \psi \quad (46)$$

and so

$$\Delta_\mu \psi = \partial_\mu \psi - \frac{i}{2} v_\mu^3 \tau^3 \psi \quad (47)$$

is a covariant derivative transforming as  $\psi$  itself in equation (46), and may be used to form invariant terms involving matter fields in the usual way [5, 7].

In the remainder of this paper we derive expressions for the covariant derivatives and metric by direct manipulation of the Pauli matrices, and remaining strictly within the chiral  $SU_2$  framework. We start by introducing a little extra calculational device by defining

$$R_{ij} = \frac{1}{2} \text{Tr}[L^{-1} \tau_i L \tau_j] \quad (48)$$

where, as before,  $i$  and  $j$  lie in the range  $1 - 3$ . Using the same formalism as in equations (21) to (29), we easily establish that

$$R_{AB} = (\delta_{AB} - n_A n_B) \cos \theta + n_A n_B, \quad (49)$$

$$R_{A3} = \varepsilon_{AB3} n_B \sin \theta = -R_{3A}, \quad (50)$$

and

$$R_{33} = \cos \theta, \quad (51)$$

where the projection operator properties are again noted. From equation (35) we see that the quantities appearing in the covariant derivatives can be expressed as

$$a_{\mu B} = (\partial_\mu M_C) a_{CB} \quad (52)$$

and

$$v_{\mu 3} = (\partial_\mu M_C) v_{C3} \quad (53)$$

where

$$a_{CB} = i \text{Tr}[\tau_B L^{-1} L_{,C}] \quad (54)$$

and

$$v_{C3} = i \text{Tr}[\tau_3 L^{-1} L_{,C}] \quad (55)$$

which we shall shortly see are particularly convenient forms. Now we return to our defining equation (21) and extract

$$\frac{-i}{2} \tau_A L = L_{,B} K_{AB} - \frac{i}{2} \lambda_{A3} L \tau_3, \quad (56)$$

and we can deduce that

$$R_{AD} = K_{AB} a_{BD} \quad (57)$$

by premultiplying by  $\tau_D L^{-1}$  and taking the trace. Since  $K_{AB}$  is non-singular, we can see that

$$a_{FD} = (K^{-1})_{FA} R_{AD}, \quad (58)$$

and hence

$$a_{FD} = (\delta_{FD} - n_F n_D) \frac{\sin \theta}{M} + n_F n_D \frac{d\theta}{dM} \quad (59)$$

follows from equations (30) and (49). Similarly, returning to equation (56) we can also deduce that

$$R_{A3} = K_{AB}v_{B3} + \lambda_{A3} \quad (60)$$

by premultiplying by  $\tau^3 L^{-1}$  and tracing. Hence we find directly that

$$v_{F3} = \frac{2}{M} \sin^2(\theta/2) \varepsilon_{FZ3} n_Z \quad (61)$$

by using equations (34) and (50). This completes our task, and we see that all the results can indeed be found from those in reference (23) by our truncation method. We do realize that we have not given a strict mathematical proof of the relationship between chiral  $\frac{SU_2 \times SU_2}{SU_2}$  and the chiral  $SU_2/U_1$  embedded in it.

It is however gratifying to see that all the results we need do come out as speculated in the truncation.

One last footnote. Just as in general relativity where tetrads or vierbeine are introduced to allow the treatment of spinors by “taking the square root of the metric”, here the unitary unimodular square root nature of  $L$  versus  $L^2$  can be exploited by introducing Killing vectors for the square root system. This concept is easier to understand in concrete form. From our defining equation (14) we can see that

$$L \tilde{g}^{-1} = h^{-1} L' \quad (62)$$

where we have inverted the equation and then applied the involutive outer automorphism  $\sim$  which reverses the signs of the generators in the group but not in the subgroup. Multiplying the respective sides of equations (14) and (62) gives

$$g L^2 \tilde{g}^{-1} = L'^2 \quad (63)$$

in which  $h$  has been eliminated thus emphasizing that the action on  $M_A$ , specified by  $K_{BA}$ , is determined by  $L^2$ . In the notation used previously we have

$$\{\tau_A, L^2\} = -2L^2_{,B} K_{AB} \quad (64)$$

as the significant part of the information. We multiply from the left by  $L^{-2} (K^{-1})_{CA}$  to see that

$$(K^{-1})_{CA} [L^{-2} \tau_A L^2 + \tau_A] = -2i L^{-2} L^2_{,C}, \quad (65)$$

then multiplying from the right by  $\frac{1}{2} \tau^B$  and tracing yields

$$(K^{-1})_{CA} [\delta_{AB} + \frac{1}{2} Tr (L^{-2} \tau_A L^2 \tau_B)] = -i Tr (L^{-2} L^2_{,C} \tau_B) \quad (66)$$

and comparison with equations (48) and (54) makes clear how the square root can be taken. We define

$$\left(k^{-1}\right)_{CA} [\delta_{AB} + R_{AB}] = iTr \left(\tau_B L^{-1} L_{,C}\right) \quad (67)$$

where the sign in taking the square root has been picked for convenience. Then equations (56) and (57) reveal that

$$\left(k^{-1}\right)_{QT} = \left(K^{-1}\right)_{QA} R_{AD} \left([1 + R]^{-1}\right)_{DT} \quad (68)$$

which the reader may enjoy confirming, reproduces the obvious inverse of  $K_{QT}$  in equation (30) when  $\theta$  is halved. This clarifies the sense of the square root. In an entirely analogous way we may write

$$\left(k^{-1}\right)_{CB} R_{B3} = iTr \left(\tau_3 L^{-1} L'_C\right) \quad (69)$$

and discover

$$\lambda_{A3} = R_{A3} - K_{AB} \left(k^{-1}\right)_{BF} R_{F3}. \quad (70)$$

Substitution of the results from equation (68) and (50) into equation (69) confirms the expression found in equation (61) for  $v_{F3}$ , while similar substitutions into equation (70) retrieve the result previously given in equation (34). The results given in this last section are not directly retrievable (as far as we know) by truncation of the results in reference [23], since the full chiral structure allowed shortcuts to be taken in that paper.

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